Multimode Resonators with a Small Fresnel Number (Lowest-Order Eigenmodes)

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The Fabry–Pèrot Interferometer, the confocal and the spherical resonator systems are investigated. The lowest-order traveling-wave type eigenmodes are calculated. Numerical values for the diffraction losses are given. The smallest diffraction losses are obtained for the general-type eigenmode of a confocal resonator system. The eigenfunctions of an open-walled resonator show a point of inflection as their characteristic feature. They are complex if the Fresnel Number is finite. When calculate over appropriate surfaces, their imaginary part, in the region close to the axis, decreases as F increases. In that region the waves resonate between the reflectors. Towards the rim of the system the imaginary part increases rapidly as do the diffracted waves associated with the imaginary part.

The multimode resonator systems with open-walled structure play a significant role for devices such as the laser. They pose a boundary value problem which cannot be solved in closed form and thus must be treated by methods of approximation. This general problem was first put forward by Fox and Li [1–4] and Boyd and Gordon [5, 6] in their noteworthy and interesting papers. Fox and Li have replaced the resonator problem by a transmission line medium consisting of a series of periodically spaced black screens into which coaxial apertures were cut. Using Huygens' Principle they have adopted a self-consistent field technique. According to this technique a normal mode consists of a field distribution over the aperture of the "equivalent" system which reproduces itself in successive transits between the mathematical surfaces. On this basis Boyd and Gordon have given an integral equation which can be solved in closed form, the eigenvalues and eigenfunctions being expressible in terms of the radial and angular wave functions in prolate spheroidal coordinates. These approaches have since been discussed or applied by several authors [7–23]. Alternatively, the problem of diffraction by an aperture can be put into the form of an angular spectrum of plane waves (Ref. [24], pp. 558 – 561, and [25]).

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Using this principle, various authors [12, 26–32] have formulated the resonator problem in terms of a consideration of a spectrum of waves bouncing back and forth between two mathematical surfaces. They have derived numerical results by a numerical iteration method [26], and by a variational method. Recently MORGAN [12, 33] suggested that the use of variational techniques may lead to results of doubtful accuracy. This is because the class of integral equations with complex, symmetric kernels does not have some of the well-known properties of integral equations with Hermitian kernels.

We shall investigate the FABRY–PÈROT Interferometer and the confocal and spherical resonator systems from a quite different point of view. The author [34] has developed a scalar resonator theory for optical frequencies which, unlike more familiar approaches, is based upon the HUYGENS–FRESNEL Principle. He has derived a general integral equation which is a solution of HELMHOLTZ's Equation within the resonator space and satisfies MAXWELL's Equations along the reflecting surfaces. This integral equation is applied to the resonator systems under investigation. We use a computational method to solve the corresponding integral equations for the eigenvalue and eigenfunction of the lowest-order eigenmodes. We then give numerical values for the diffraction losses as a function of the FRESNEL Number and demonstrate characteristic features of eigenfunctions which describe resonator systems with open-walled structure. The eigenfunctions are complex if the FRESNEL Number is finite. Hence, the reflectors do not, in general, coincide with a surface of constant phase, but at the most near the axis of the system.

It may already be remarked here that our results are different from the corresponding ones of other investigators. We believe, however, that the analysis and the results presented in this paper are in a complete agreement with classical electrodynamics and with the experimental results reported in recent literature.

1. General Formulation

We investigate resonator systems with two equal spherical reflectors a distance d apart. These reflectors are assumed to be perfectly conducting, to be equal in size and to be square when viewed in the z-direction. All dimensions are large compared to the wavelength, and the field is assumed to be linearly polarized in the y-direction. We define a traveling-wave type eigenmode [34] as an energy distribution which, when launched from a mathematical surface between the reflectors, reproduces itself within a constant factor on the same surface after N bounces. The periodicity N is determined by the geometry of the resonator system and may be obtained with the aid of geometrical optics [Ref. [34], Eq. (6)]

$$N = \frac{2\pi}{\arccos(1 - d/R)}, \tag{1}$$

where R is the common radius of curvature of the reflectors and d their separation.

The integral equations describing the resonator systems under investigation are deduced from a general one [Ref. [34], Eq. (4)]. This integral equation represents a solution of HELMHOLTZ's Equation in the resonator space and satisfies MAXWELL's Equations along the reflecting surfaces. The systems with which we shall ultimately deal are described by very involved integral equations. These may, however, be simplified considerably as pointed out in [34]. We use Cartesian coordinates. Since only small angles are involved, we neglect the variation of the inclination factors over the apertures. The distances s, which a paraxial wave travels between arbitrary points on two surfaces, vary only by a small amount for small apertures. Thus, they may be replaced in terms of d except in the exponential phase term of the general kernel [Ref. [34], Eq. (5)]. No cross products between x and y appear in any term of the corresponding integral equations. Hence, assuming that the eigenfunctions and eigenvalues can be represented in product form, these integral equations

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can be separated and written as a product of two identical equations. One equation depends only upon x and the other one only upon y. Henceforth, we shall merely investigate the integral equation dependent on x and use the same results for the y-dependent one. Thus, we assume that the field pattern of the pq-th eigenmodes is given by

$$E_y(x, y) = E_0 \, X_p(x) \; Y_q(y) \tag{2}$$

on the mathematical surface S^{m}, where E_0 is a constant amplitude factor, and that the corresponding eigenvalues are given by

$$\varkappa_{pq} = \varkappa_p \, \varkappa_q \, . \tag{3}$$

In some cases if $N > 1$ the calculations may be simplified even further by taking into consideration the symmetry of the system under investigation. This is true especially, if an eigenmode can be subdivided into an integral number γ of identical sections, as we shall see later. Then, if

$$\varrho = N/\gamma \tag{4}$$

is an integer we may write the x-dependent integral equation in a general form

$$\varkappa_p \, X_p(x_{\varrho+1}) = \int_{\alpha}^{\beta} K(x_{\varrho+1}, x_0) \; X_p(x_0) \; \mathrm{d}x_0 \, . \tag{5}$$

The corresponding kernel for a specific system under investigation will be stated below. The integration in (5) has to be performed over a "mathematical surface", i. e. a surface on which no particular boundary conditions require consideration. In fact, this integration should be extended to infinity. We must, however, restrict it to a finite range so that we can apply a computational method to solve (5) numerically, as we shall discuss later.

The reflection losses result from absorption in the reflectors and from transmission through them. All remaining losses of a free space resonator system we term "diffraction" losses. Thus, the diffraction losses with which we are ultimately concerned result from the finite aperture of the reflectors and from imperfections in their "flatness." From energy balance considerations we easily see that the fractional diffraction loss of the pq-th eigenmodes is given by the relation

$$\alpha_{\mathrm{D}} = 1 - |\varkappa_{pq}|^{2\gamma} = 1 - |\varkappa_p \, \varkappa_q|^{2\gamma} \, . \tag{6}$$

If $p = q$, this relation reduces to

$$\alpha_{\mathrm{D}} = 1 - |\varkappa_p|^{4\gamma} \tag{6 a}$$

[35] H. Heffner, Stanford University, private communication. — See also H. K. V. Lotsch, Z. Naturforschg. **19 a.** 1438 [1964].

Heffner [35] has proposed a self-consistent field technique which consists of a superposition on both reflectors of the various patterns belonging to an eigenmode. The losses per transit of such a self-consistent field distribution are approximately identical with those obtained for the pq-th traveling-wave type eigenmode.

1.1 Fabry–Pèrot Interferometer

The Fabry–Pèrot Interferometer consists of two parallel plane mirrors a distance d apart. It may be considered a limiting case of a resonator system with spherical reflectors whose radii of curvature are infinitely large. The formulation of the problem is illustrated in Fig. 1. The plane of symmetry is chosen as the mathematical surface. As a quasi steady-state solution we postulate a pattern which, when launched from this plane, is reproduced in the same plane after one reflection by, say, the right mirror. If x' is small compared to d we can show that [36]

$$s_{01} + s_{12} \approx d + \frac{(x_2' - x_0')^2}{2\,d} + \frac{2}{d}\left(x_1' - \frac{x_2' + x_0'}{2}\right)^2 . \tag{7}$$

Using this relation we may easily derive the corresponding kernel for the x-dependent integral equation (5) with $\varrho = 1$ which from [Ref. [34], Eq. (5)] is

$$K(x_2, x_0) = 2\,F \exp\{i\,k\,d/2\} \cdot \exp\{i\,\pi\,F(x_2 - x_0)^2\} \cdot \int_{-1}^{+1} \exp\left[i\,4\,\pi\,F\left(x_1 - \frac{x_2 + x_0}{2}\right)^2\right] \mathrm{d}x_1 \, . \tag{8}$$

We have introduced the dimensionless variable $x = x'/a$, $2\,a$ is one side of a reflector, and $k = 2\,\pi/\lambda$ is the propagation constant for the wavelength λ. The Fresnel Number F is defined by

$$F = a^2/(\lambda\,d) \, . \tag{9}$$

The fractional diffraction loss of the pq-th eigenmode is obtained from (6) with $\gamma = 1$.

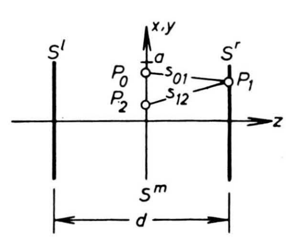

Fig. 1. Geometry of the Fabry–Pèrot Interferometer. S^{l} and S^{r} represent the left and right mirror, respectively.

[36] Note that here and in (12) the primes tell x from x/a, but later on when this distinction is obvious they are omitted.

1.2 Confocal Resonator System

The confocal-type resonator consists of two spherical reflectors separated by their common radius of curvature. As pointed out in [34] the confocal resonator system possesses two different sets of eigenmodes. This we have schematically illustrated in Figs. 2 and 3 using geometrical optics. For ease of reference we will call them general-type (Fig. 2) and V-type (Fig. 3) eigenmodes. For the formulation of the corresponding integral equations we choose the plane of symmetry as the mathematical surface.

1.21 General-Type Eigenmodes

We note from inspecting Fig. 2 that a pattern is reproduced, but inverted, after half the periodicity of four bounces, ($N=4$). Because of the symmetry of the system the eigenfunctions are either even or odd with respect to the z-axis. Hence, we postulate a self-consistent field distribution for such a section of an eigenmode, ($\gamma=2$). The x-dependent integral equation is obtained from (5) with $\varrho=2$ and we state the corresponding kernel from [Ref. [34], Eq. (18)]

$$K(x_3, x_0) = 2\,F^{3/2} \exp\{i(k\,d+\pi/4)\} \cdot \exp\{i\,2\,\pi\,F(x_0^2-x_3^2)\} \cdot \int_{-1}^{+1} \exp[-i\,4\,\pi\,F\,x_1(x_3+x_0)] \cdot \int_{-1}^{+1} \exp[i\,\pi\,F(x_2-2\,x_3-x_1)^2]\;dx_2\,dx_1 . \quad (10)$$

The fractional diffraction loss of the pq-th eigenmode is obtained from (6) with $\gamma=2$.

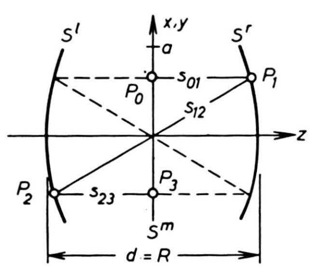

Fig. 2. Illustrating the general-type eigenmode of a confocal resonator system.

[37] H. Lotsch, IEEE Trans. Microwave Theory Techn. **MTT-12**, 482 [1964].

1.22 V-Type Eigenmodes

We note from inspecting Fig. 3 that a pattern is reproduced from bounce to bounce. Thus, four of these sections compose an eigenmode, ($N=4$, $\gamma=4$). We postulate a self-consistent field distribution. The x-dependent integral equation is given by (5) with $\varrho=1$. The corresponding kernel is given in [37]

$$K(x_2, x_0) = \frac{F}{2} \exp\{i\,k\,d/2\} \exp\{-i\,\pi\,F\,x_2\,x_0\} \cdot \int_{-2}^{+2} \exp\{i\,\pi\,F(x_1-x_2-x_0)^2/2\}\;dx_1 . \quad (11)$$

The fractional diffraction loss of the pq-th eigenmode is obtained from (6) with $\gamma=4$.

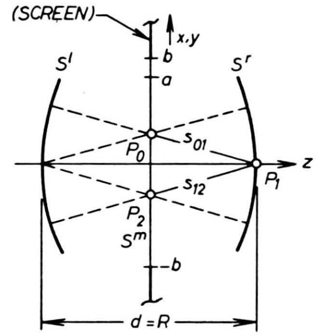

Fig. 3. Illustrating the V-type eigenmode of a confocal resonator system and the discussion of Chapter 6.

1.3 Spherical Resonator System

The spherical resonator system consists of two spherical reflectors separated by twice their common radius of curvature, Fig. 4. As pointed out in [38] the concentric spherical surfaces midway between the reflectors and the center of the system coincide with a surface af constant phase in the geometrical-optics solution. Therefore, let us use this feature in formulating the spherical problem. We postulate a quasi steady-state solution in the following way. We require that a pattern when launched from S_0^{m} be reproduced on the mirror surface S_2^{m} after having been reflected by the right reflector, ($N=2$, $\gamma=2$). The x-dependent integral equation is given by (5)

[38] H. Lotsch, Z. Angew. Phys. **18**, 241 [1964].

with $\varrho = 1$. If x' is small compared to d we can show that

$$s_{01} + s_{12} \approx d - \frac{2}{d}\,(x_0{}' + x_2{}')^2 + \frac{2}{3\,d}\,(x_1{}' - 3\,x_0{}' - x_2{}')^2 . \tag{12}$$

Using this relation with $x = x'/a$ we deduce the corresponding kernel from [Ref. [34], Eq. (5)]

$$K(x_2, x_0) = \frac{F}{\sqrt{3}} \exp\{i(k\,d/2 + \pi/4)\} \cdot \exp\{-i\,\pi\,F(x_0 + x_2)^2\} \cdot \int_{-2}^{+2} \exp[i\,\pi\,F(x_1 - 3\,x_0 - x_2)^2/3]\,\mathrm{d}x_1 . \tag{13}$$

The fractional diffraction loss of the $p\,q$-th eigenmode is obtained from (6) with $\gamma = 2$.

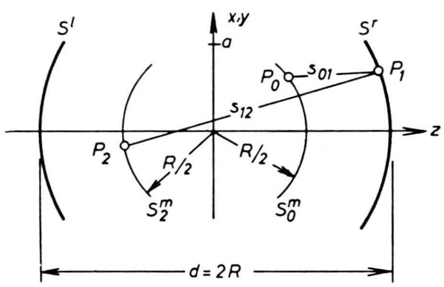

Fig. 4. Illustrating the formulation of kernel (13) for the spherical resonator system.

1.4 Discussion of these Integral Equations

The integral equations as stated above are consistent with the geometrical-optics solution in the limit as $F \to \infty$. Applying the method of stationary phase [34], we easily convince ourselves that the FRESNEL Integrals of (8), (10), (11), and (13) account for the law of reflection. Furthermore, as $F \to \infty$ the diffraction losses vanish and the phase over the mathematical surfaces becomes constant. Note that the kernels (11) and (13) have been transformed by $x \to x/2$. Thus, the eigenfunctions of all four cases are restricted to the range $(-1, 1)$ in the geometrical-optics solution. In this limit the eigenfunctions are identical with zero outside of that range. As F decreases these eigenfunctions extend beyond the minimum range $(-1, 1)$. For the purpose of comparing these four cases, we must find a compromise for the range (α, β) of (5), which, correctly speaking, should be $(-\infty, +\infty)$. It seems reasonable, especially from the standpoint of computation, to limit this range to a 20 percent increase over its minimum range in the geometrical-optics solution, hence $\alpha = -1.2$ and $\beta = 1.2$. Thus, we assume that the contribution from the range outside of $(-1.2, 1.2)$ can be neglected. By doing so we introduce an inherent error into the numerical results which we should keep in mind especially if the FRESNEL Number is small.

These integral equations look too involved to expect solutions in terms of ordinary functions. Therefore, we use an iterative technique to obtain numerical values for the diffraction losses and the eigenfunctions. We restrict ourselves to the lowest-order eigenmode in each case, $(p = q \equiv 0)$. Their spatial resonance spectrum is determined by the geometrical phase shift

$$N(1+\delta)\,k\,d = m\,2\,\pi\,, \qquad m = \text{integer} \tag{14}$$

where $\delta = 1$ if $N = 1$ else 0. For convenience we redefine the eigenvalue $\varkappa$ of (5) to absorb the constant phase factors of the various kernels (8), (10), (11), and (13).

$$\sigma_p = \varkappa_p\, e^{-i(\quad)} . \tag{15}$$

We have ignored the extra phase shift to compensate for the shift due to the formulation of the integral equation [34]. In the analysis presented we shall only compute the absolute value of the eigenvalues and thus are not concerned with a pure rotation.

2. Numerical Solution

There is an obvious analogy between functions and vectors. We may consider $X(x)$ to be a vector with an infinite dimensionality; $X(x)$ is the x-th component of X. Thus,

$$\sigma\,X(x_i) = \int K(x_i, x_j)\,X(x_j)\,\mathrm{d}x_j \tag{16}$$

is roughly the same as the matrix equation

$$\sigma\,X_i = \sum_j W_j\,K_{ij}\,X_j = \sum_j M_{ij}\,X_j . \tag{17}$$

The weight coefficient W is introduced because the initial integral is approximated by a finite number of subdivisions; it depends upon the particular quadrature formula used. GAUSS' method of mechanical quadrature has an advantage over most methods of numerical integration in that it requires fewer ordinate computations. GAUSS' classical result states that, for the range $(-1, 1)$, the "greatest" accuracy with n ordinates is obtained by choosing the corresponding abscissa at the zeros ξ_1, ξ_2, ξ_3, ...,

ξ_n of the LEGENDRE Polynomials $P_n(\xi)$. With each ξ_j is associated a weight coefficient W_j such that an integral with arbitrary limits, α and β, is approximately given by

$$\int_{\alpha}^{\beta} f(\xi)\, d\xi \approx \frac{\beta-\alpha}{2} \sum_{j=1}^{n} W_j\, f\left(\xi_j \frac{\beta-\alpha}{2} + \frac{\beta+\alpha}{2}\right). \quad (18)$$

Tables of the zeros of the LEGENDRE Polynomials and the associated weight coefficients for GAUSS' quadrature formula are readily available [39–41].

A Burroughs B 5000 Computer was programmed [42] to solve (17) by an ordinary iterative technique. An initial vector X_j was arbitrarily chosen. The iterative procedure was terminated when the relative change of the absolute value of σ became smaller than a specified limit. The convergence is determined by the ratio [41] $|\sigma_1|/|\sigma_0|$. This ratio tends to unity as F increases, and thus it limits the applicability of this iterative technique. In order to extend this limit the actual procedure was carried out using the fourth power of the matrix M of (17).

The various kernels are complex. They have been separated in real and in imaginary part as shown in the Appendix. The iteration procedure was carried out on a complex basis using library procedures [43, 44]. The FRESNEL Integrals were generated by a standard procedure [45].

3. Diffraction Losses

We have plotted the fractional diffraction losses of the lowest-order eigenmodes as a function of the FRESNEL Number F in Fig. 5. For $F \to 0$ all the curves converge into the point "100 per cent". The FABRY–PÈROT Interferometer shows lower losses in approaching this limit. As F increases the functional dependence of each curve changes significantly. As expected, the confocal resonator system has the largest quality constant. The general-type eigenmode clearly has the lowest losses. To the author's knowledge this type of eigenmode has not yet been treated in the literature of quantum electronics. As is pointed out in [34] it cannot be obtained by the more familiar approaches to the boundary value problem under investigation. It seems questionable to the author whether in those techniques the boundary conditions subject to MAXWELL's Equations are satisfied [18, 45a]. It is interesting to note that several authors [3, 5] have reported diffraction losses for a confocal-type resonator whose magnitudes are of the order of the ones obtained for the general-type eigenmode. We, however, direct attention to the fact that according to their formulation those values correspond to the ones of the V-type eigenmode. But we have obtained considerably higher losses for the V-type eigenmode.

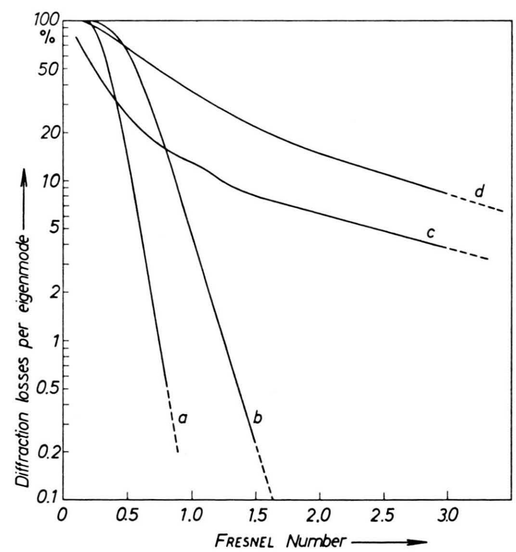

Fig. 5. Diffraction losses per eigenmode versus FRESNEL Number F. a) General-type eigenmode of a confocal resonator system, b) V-type eigenmode of a confocal resonator system, c) FABRY–PÈROT Interferometer, d) Spherical resonator system.

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[42] The author wishes to thank Mr. R. W. COLE and Mr. P. BUSINGER, Computation Center, Stanford University, for invaluable aid in programming techniques.

[43] J. H. WELSCH, Complex Matrix Multiplication, Computer Science Library, Stanford University, Stanford, California.

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[45] H. LOTSCH and M. GRAY, FRESNEL Integrals, Communication of the ACM **7**, 660 [1964].

Let us illuminate the present results from a different point of view. The general-type eigenmode corresponds to a telescopic system [46], in the optical analogy of a periodic sequence of thin lenses. The plane front of a traveling wave is inverted into a convergent spherical wavefront at a reflector or in a lens. Correspondingly, a divergent spherical wave is transformed into a plane wave. Hence, a traveling wave when launched from the plane of symmetry returns to the same plane with a plane wavefront. On the other hand the V-type eigenmodes require spherical wavefronts in the space between the reflectors [46a]. Such wavefronts are established if the waves predominantly resonate in that space. This, however, is true only if the losses are small. From a similar point of view we may expect lower losses in the case of the FABRY–PÈROT Interferometer than in that of the spherical resonator.

We direct attention to the fact that other investigators have calculated the diffraction losses per transit. In contrast to their method we have defined the losses per eigenmode, i. e. for N transits. Thus, the present values are identical with the ones obtained by a self-consistent field technique according to HEFFNER (see Chapter 1).

In Section 1.4 we have pointed out that a finite range (α, β) of integration may affect the numerical results. This inherent error depends upon the amplitude of the eigenfunctions outside of this range of integration. To get a feeling for its order of magnitude we have recalculated several parameters for a range being 40 percent larger than the minimum range in the geometrical-optics solution. For ease of comparison we have summarized these results in Table 1. In agreement with the discussion above the general-type eigenmode is less sensitive than the V-type eigenmode; a similar result holds for the eigenmode of both the FABRY–PÈROT Interferometer and the spherical resonator system. Some additional information may be obtained from Chapter 6.

		F	Diffraction losses per eigenmode in per cent for $(\alpha, \beta) =$ $(-1.2, 1.2)$	$(-1.4, 1.4)$
Fabry-Perot Interferometer		0.5	26.224	30.876
		2.5	4.574	4.471
Confocal Resonator System	General-type eigenmode	0.25	86.468	84.821
		0.5	14.841	13.766
	V-type eigenmode	0.5	66.153	44.054
		1.0	5.417	1.895
Spherical Resonator System		0.5	67.607	59.968
		2.5	10.957	8.926

Table 1. Diffraction losses for two different ranges of integration over the mathematical surface.

4. Eigenfunctions (Real Part)

We have computed the lowest-order eigenfunctions of the four cases under investigation. They are normalized such that, close to the axis of the system, the imaginary part vanishes and the real part is unity. As pointed out in [34] the eigenfunctions are purely real over the appropriate mathematical surfaces as $F \rightarrow \infty$, and HELMHOLTZ's Equation is solved by a pure standing wave. Hence, for a finite F-value we may assume that the real part of an eigenfunction describes the standing-wave type component in the solution of HELMHOLTZ's Equation. The traveling-wave type component, i. e. the diffracted wave, is associated with the imaginary part. Therefore, we have plotted the real part of the eigenfunctions in Fig. 6, since only this part is associated with the waves resonating between the reflectors. We note that the abscissa of Fig. 6 are merely extended over the minimum range in the geometrical-optics solution.

As F increases the amplitude of the eigenfunctions away from the axis decreases. Moreover, we recognize the connections with Fig. 5. The energy of the general-type eigenmode is confined in a region close to the axis, and thus the diffraction losses are small. We direct attention to the fact that especially as F increases the eigenfunctions reveal a point of inflection. We believe that this is a characteristic feature of an eigenfunction describing an open-walled resonator system. Hence, the lowest-order

[45a] H. LOTSCH, Paper Q 6, Winter Meeting Amer. Phys. Soc., Berkeley, Calif., Dec. 1964. – H. LOTSCH, Bull. Amer. Phys. Soc. (II) **9**, 729 [1964].

[46] P. DRUDE, The Theory of Optics, Dover Publications, Inc., New York 1959, pp. 26—30.

[46a] H. LOTSCH, On the Model of the "Equivalent" Confocal Resonator System, IEEE Trans. Electron Devices, submitted for publication.

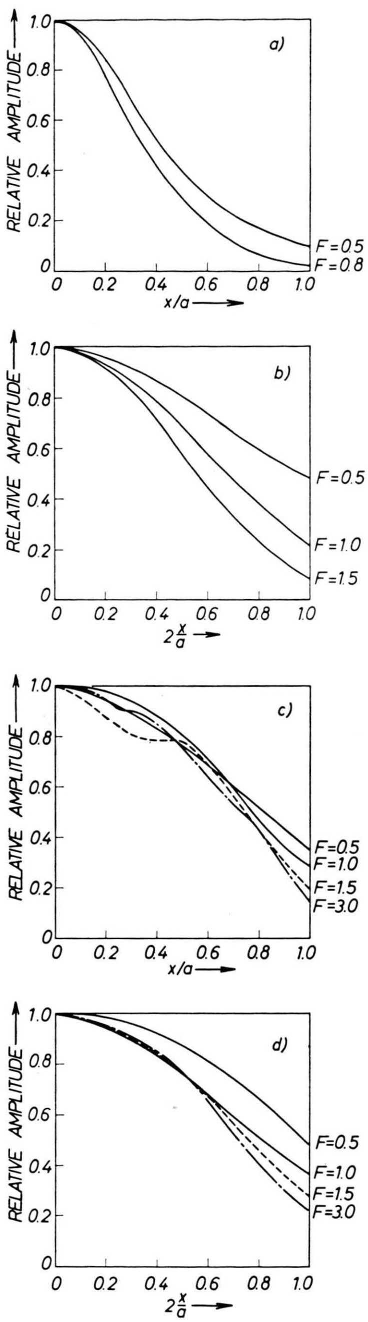

Fig. 6. Real part of the lowest-order eigenfunction for different F-values. a) General type eigenmode of a confocal resonator system, b) V-type eigenmode of a confocal resonator system, c) Fabry–Pèrot Interferometer, c) Spherical Resonator system.

eigenfunction of a Fabry–Pèrot Interferometer is not a cosine function as widely assumed. As F increases the energy tends to concentrate itself more and more around the axis of the system since the eigenfunctions shrink, as we also see from the graphs in [5]. The author [37] has demonstrated that, for a large Fresnel Number and near the axis of the system, the V-type eigenmodes can be described in terms of the one-dimensional harmonic oscillator wave functions. Thus, considering the corresponding lowest-order eigenfunction of that asymptotic solution, we recognize the connection between that analysis and the present computations.

When observing the mode pattern of a laser we observe the intensity which is proportional to the eigenfunction squared. Hence, we easily convince ourselves that for a large Fresnel Number the bright spot being observed is much smaller than the size of the mirror if only a few orders oscillate. Let us consider the general-type eigenmode with $F=0.8$ as an example. The intensity drops to 1/2 of its maximum at a distance of about $x=a/5$ from the axis.

In a manner similar to that of Fox and Li [3] we have also found that for some values of F the eigenfunctions of a Fabry–Pèrot Interferometer are wavy, however, only close to the axis. It is interesting to note that for those F values the curve describing the diffraction losses, Fig. 5, shows a slight hump.

5. Phase of the Eigenfunctions

As discussed in the previous chapter, the imaginary part of the eigenfunction is associated with the diffracted waves. It vanishes as $F \rightarrow \infty$, i. e., in the geometrical-optics solution. We have obtained the significant result that all eigenfunctions belonging to a traveling-wave type eigenmode are, in general, complex [47]. We have computed the phase variation over the mathematical surface and plotted it versus

[47] Soohoo [14–17] who has discussed his work with George [48] has determined that the phase is constant over spherical reflectors only when spaced confocally. We direct attention to a postulate of the classical electrodynamics. If and only if the reflectors coincide with a surface of constant phase does any physical solution of Helmholtz's Equation which satisfies Maxwell's Equation subject to these boundary conditions have the nature of the pure standing-wave. A pure standing-wave does not, however, yield losses even though the resonator has no side-walls.

[48] N. George, Electrical Engineering Department, California Institute of Technology, Pasadena, California (private communications during the period of the academic years 1961–62 and 1962–63 when he was the research adviser of the author).

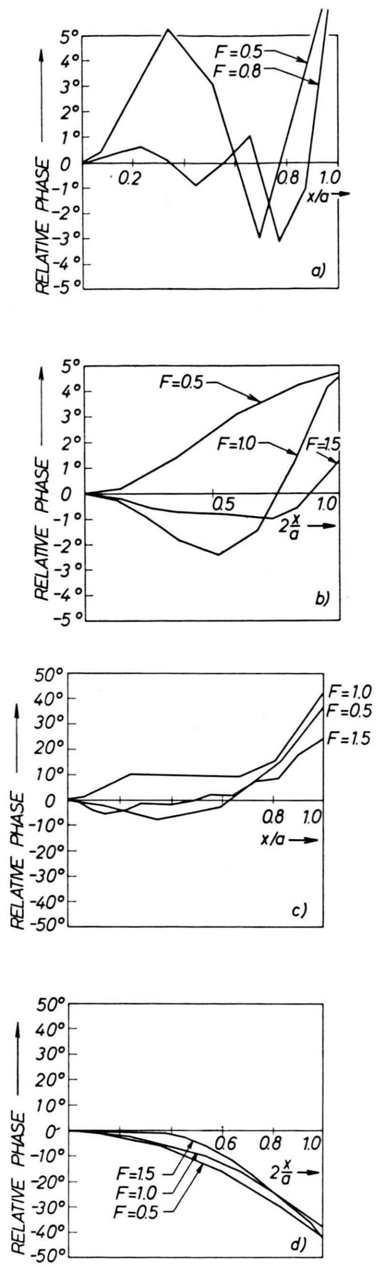

Fig. 7. Phase variation of the lowest-order eigenfunctions for different F-values. a) General-type eigenmode of a confocal resonator system, b) V-type eigenmode of a confocal resonator system, c) FABRY–PÈROT Interferometer, d) Spherical resonator system. The scales of the phase coordinates differ by a factor of 10 in (a), (b) relative to (c), (d).

the radial distance in Fig. 7. The phase is small near the axis and increases rapidly towards the end points. This behaviour is in excellent agreement with the asymptotic solution derived by the author [37].

When F increases the phase decreases close to the axis. In this region the waves resonate between the reflectors. Thus, in agreement with diffraction theory the diffracted waves predominantly occur close to the rim of the system. We direct attention to the different scales of Figs. 7 a, b and Figs. 7 c, d. With the ten-fold enlarged scales of Figs. 7 a, b we wish to emphasize that in the case of the confocal-type resonator the imaginary part does not vanish in the central region. It is, however, smaller than in the cases of the FABRY–PÈROT Interferometer and of the spherical resonator system, since the diffraction losses of the confocal-type resonator are smaller.

6. Aperture Limited Confocal-Type Resonator

It is interesting to investigate how an aperture in the plane of symmetry affects the general-type and the V-type eigenmode, (Fig. 3). For both modes we consider a system identical in the mechanical dimensions. Thus, we must transform the kernel (11) by $(x/2 \to x)$ and obtain

$$K(x_2, x_0) = 2\,F\,\exp\{i\,k\,d/2\}\,\exp\{-i\,4\,\pi\,F\,x_2\,x_0\} \cdot \int_{-1}^{+1} \exp\{i\,2\,\pi\,F\,(x_1 - x_2 - x_0)^2\}\;dx_1\,. \tag{19}$$

The integral equation is again given by (5) with $\varrho = 1$. The interval (α, β) is now specified to be $(-b, b)$. For $F = 0.5$ and 0.75 we have computed

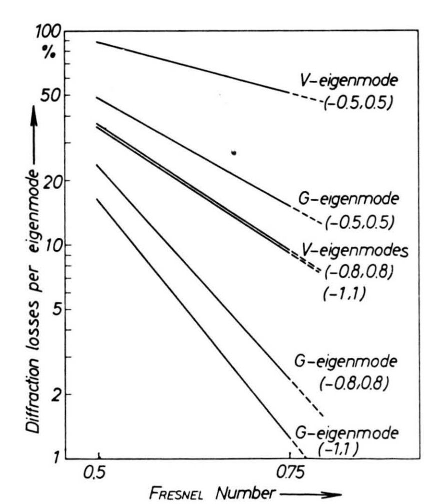

Fig. 8. Aperture limited confocal resonator system for three different sizes of the aperture relative to the size of the mirrors: $(-1, 1)$, $(-0.8, 0.8)$ and $(-0.5, 0.5)$. G stands for general-type eigenmode and V for V-type eigenmode.

the diffraction losses for the three different aperture sizes ($b=1$; $b=0.8$; and $b=0.5$). The results are plotted in Fig. 8. For convenience we have connected the two points by a straight line. The general-type eigenmode depends strongly upon the aperture size since this mode completely fills up the whole volume of the resonator. In comparison with Fig. 5, however, we note the important fact that its losses are scarcely affected by an aperture of the same size as the reflectors, i. e. ($b \approx 1$). For the V-type eigenmode the aperture sizes ($b=1$ and $b=0.8$) correspond to ranges (α, β) of (5) which are respectively larger by 100 and 60 percent than the minimum range in the geometrical-optics solution. Hence, they yield about the same values for the diffraction losses and most likely they are more accurate than the corresponding ones in Fig. 5. These results are in good agreement with the discussion of Chapter 3.

7. Conclusion

We have investigated the FABRY–PÈROT Interferometer, the confocal and the spherical resonator systems. The lowest-order traveling-wave type eigenmodes are computed for small values of the FRESNEL Number. We have given numerical values for the diffraction losses, the real part of the eigenfunctions, and the phase of the eigenfunctions. This investigation has led to the following conclusions:

1. The confocal resonator system has the highest quality-constant. The general-type eigenmode shows the lowest diffraction losses. The diffraction losses of the V-type eigenmode are considerably higher than the values reported by other investigators.

2. The FABRY–PÈROT Interferometer shows lower diffraction losses than the spherical resonator system. These losses are much larger than the ones obtained for the confocal-type resonator.

3. The eigenfunctions of a traveling-wave type eigenmode describing a resonator system with open-walled structure are complex. If computed over an appropriate mathematical surface their real part describes the waves resonating between the reflectors. The imaginary part is associated with the traveling-wave type component in the solution of HELMHOLTZ's Equation and thus with the diffracted waves. As F increases the imaginary part decreases close to the axis but rapidly increases towards the rim of the resonator system.

4. The eigenfunctions describing an open-walled resonator system reveal a point of inflection as a characteristic feature. Hence, as F increases the energy tends to concentrate itself more and more around the axis of the system and the eigenfunctions shrink. The intensity of a mode pattern is usually observed. Thus, it seems that this feature explains the experimental fact that the observable mode pattern of a laser covers an area much smaller than a mirror.

5. The side-walls of a laser tube or a laser rod, if they have about the same diameter as the confocally spaced reflectors, should scarcely affect the diffraction losses when the FRESNEL Number is not too small.

Acknowledgements

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Appendix

Real and Imaginary Parts of (17) for the Various Cases

The FRESNEL Integrals, (Ref. [24], pp. 429–432, and [49]), are defined by

$$C(w) = C(w_1) + C(w_2) = \int_0^w \cos(\tfrac{1}{2}\pi \cdot t^2)\, dt\,, \quad \text{(A 1)}$$

$$S(w) = S(w_1) + S(w_2) = \int_0^w \sin(\tfrac{1}{2}\pi \cdot t^2)\, dt\,, \quad \text{(A 2)}$$

$$F(w) = C(w) + i\,S(w)\,. \quad \text{(A 3)}$$

We note that $C(-w) = -C(w)$ and $S(-w) = -S(w)$.

(a) *Fabry–Pèrot Interferometer, Eq. (8)*

We use the shorthands

$$w_1 = \sqrt{8}\,F[1-(x_2+x_0)/2]$$

[49] E. JAHNKE, F. EMDE, and F. LOESCH, Tables of Higher Functions, McGraw-Hill Book Co., Inc., New York, Toronto, London 1960, pp. 26–36.

and $w_2 = \sqrt{8\,F}\,[1 + (x_2 + x_0)/2]$.

Finally the real part M' of the matrix M is

$$M' = \sqrt{\frac{F}{2}}\, W(x_0)\,[C(w)\,\cos \pi F (x_2 - x_0)^2 - S(w)\,\sin \pi F (x_2 - x_0)^2] \quad \text{(A 4 a)}$$

and the imaginary part M'' of the Matrix M is

$$M'' = \sqrt{\frac{F}{2}}\, W(x_0)\,[S(w)\,\cos \pi F (x_2 - x_0)^2 + C(w)\,\sin \pi F (x_2 - x_0)^2] \,. \quad \text{(A 4 b)}$$

(b) *Confocal Resonator System, General-Type Eigenmodes, Eq. (10)*

$$w_1 = \sqrt{2\,F}\,(1 - x_1 - 2\,x_3)\,, \quad w_2 = \sqrt{2\,F}\,(1 + x_1 + 2\,x_3)\,;$$

$$I' = \int_{-1}^{+1} [C(w)\,\cos 4\,\pi F\,x_1 (x_3 + x_0) + S(w)\,\sin 4\,\pi F\,x_1 (x_3 + x_0)]\,\mathrm{d}x_1\,,$$

$$I'' = \int_{-1}^{+1} [S(w)\,\cos 4\,\pi F\,x_1 (x_3 + x_0) - C(w)\,\sin 4\,\pi F\,x_1 (x_3 + x_0)]\,\mathrm{d}x_1\,;$$

$$M' = \sqrt{2}\,F\,W(x_0)\,[I'\,\cos 2\,\pi F (x_0^2 - x_3^2) - I''\,\sin 2\,\pi F (x_0^2 - x_3^2)]\,, \quad \text{(A 5 a)}$$

$$M'' = \sqrt{2}\,F\,W(x_0)\,[I''\,\cos 2\,\pi F (x_0^2 - x_3^2) + I'\,\sin 2\,\pi F (x_0^2 - x_3^2)]\,. \quad \text{(A 5 b)}$$

(c) *Confocal Resonator System, V-Type Eigenmodes, Eq. (11)*

$$w_1 = \sqrt{F}\,(2 - x_2 - x_0)\,, \quad w_2 = \sqrt{F}\,(2 + x_2 + x_0)\,;$$

$$M' = 0.5\,\sqrt{F}\,W(x_0)\,[C(w)\,\cos(\pi F\,x_2\,x_0) + S(w)\,\sin(\pi F\,x_2\,x_0)]\,, \quad \text{(A 6 a)}$$

$$M'' = 0.5\,\sqrt{F}\,W(x_0)\,[S(w)\,\cos(\pi F\,x_2\,x_0) - C(w)\,\sin(\pi F\,x_2\,x_0)]\,. \quad \text{(A 6 b)}$$

(d) *Spherical Resonator System, Eq. (13)*

$$w_1 = \sqrt{2\,F/3}\,(2 - 3\,x_0 - x_2)\,, \quad w_2 = \sqrt{2\,F/3}\,(2 + 3\,x_0 + x_2)\,;$$

$$M' = \sqrt{F/2}\,W(x_0)\,[C(w)\,\cos \pi F (x_0 + x_2)^2 + S(w)\,\sin \pi F (x_0 + x_2)^2]\,, \quad \text{(A 7 a)}$$

$$M'' = \sqrt{F/2}\,W(x_0)\,[S(w)\,\cos \pi F (x_0 + x_2)^2 - C(w)\,\sin \pi F (x_0 + x_2)^2]\,. \quad \text{(A 7 b)}$$